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Engineering Notes

Practical Method for Optimization of Low-Thrust Transfers

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Nomenclature

$\mathbf{e}$	=	thrust direction
H_σ	=	switching function
I_{SP}	=	specific impulse
J	=	cost function
m	=	mass
$\mathbf{r}$	=	position
$\mathbf{T}$	=	thrust vector
T_{MAX}	=	maximum thrust
t	=	time
$\mathbf{v}$	=	velocity
$\mathbf{x}$	=	state vector
ε	=	smoothing parameter
λ_m	=	mass adjoint or costate
$\boldsymbol{\lambda}\mathbf{r}$	=	position adjoint or costate
$\boldsymbol{\lambda}\mathbf{v}$	=	velocity adjoint or costate
μ	=	gravity parameter
σ	=	thrust modulation parameter

I. Introduction

THE mission analyst usually needs to compute optimal spacecraft trajectories using an electrical propulsion system. Typical low-thrust mass-optimal transfers with control constraints have a bang–bang structure [1]. It means that the optimal thrust switches instantaneously from maximum to zero or vice versa several times during the transfer. The location of the switching points is not obvious for most transfers of interest. Thus, an efficient method to compute optimal low-thrust trajectories without any knowledge of the thrusting structure is necessary.

Optimization methods are generally divided into direct and indirect methods. Direct methods discretize the trajectory and the control to get a finite, although large, set of optimization parameters, which allows solving the constrained optimization problem with nonlinear programming (NLP) procedures [2]. Many methods have been presented to compute good initial guesses. Shape-based methods are quite popular and consist of assuming a certain shape of the trajectory and finding the parameters that fulfill the boundary conditions [3,4].

Indirect methods use the calculus of variations to derive optimality (necessary) conditions. These methods increase the order of the system introducing the costates or adjoints [1]. The initial infinite-dimension optimization problem is reduced to a finite-dimension problem called the two-point boundary value problem (TPBVP). The TPBVP is a system of nonlinear equations derived from the first-order optimality conditions.

The main benefits of the indirect methods compared with the direct methods are that the entire trajectory and control are described with a reduced set of parameters and that no a priori knowledge of the thrust profile is needed. The drawbacks are that they need complete reformulation to adapt to problem changes and they are very sensitive to the small variations in the optimization parameters.

Hybrid methods are less known and aim to get the benefits from direct and indirect methods [5]. These methods parameterize the control according to the optimality conditions as an indirect method, and they solve the optimization problem (not a TPBVP) with a NLP as a direct method. In some specific trajectory design problems, hybrid methods are very convenient, because the thrusting structure is known, and the initialization of the initial adjoint values is greatly simplified [6–8].

The optimal transfer with the fixed time, the specified initial state, and the specified final position and velocity is the basic problem to analyze interplanetary trajectories. A method for use in engineering applications shall compute the optimal transfer without any initial guess and is fast enough to permit extensive analyses. An indirect method is preferred, since the optimal control is derived in closed form as a function of a few parameters, and no a priori knowledge of the switching structure is required. Recently, methods to solve the TPBVP with shooting algorithms perform an extensive exploration of the search space [9–11] (often called the multistart technique). This Note follows this trail, introducing a new transformation that reduces the computational time and increases the radius of convergence of the TPBVP solution process.

II. Low-Thrust Trajectory Optimization

The low-thrust interplanetary transfer problem is formulated as an optimal control problem. Since the algorithm is intended for an engineering application, only the first-order optimality conditions will be considered. The objective is to maximize the final mass of the spacecraft. The cost function to be minimized is presented in Eq. (1):

$$J = \phi[\mathbf{x}(t_f)] = -m(t_f) \tag{1}$$

The dynamics of the state vector are described by the nonlinear differential equation given in Eq. (2), where the specific impulse is constant, and the thrust magnitude is constrained [Eq. (3)]:

$$\dot{\mathbf{x}} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\begin{Bmatrix} \mathbf{r} \\ \mathbf{v} \\ m \end{Bmatrix} = \begin{Bmatrix} \mathbf{v} \\ -\mu\mathbf{r}/r^3 + \mathbf{T}/m \\ -\|\mathbf{T}\|/I_{SP}g_0 \end{Bmatrix} \tag{2}$$

The control function is the thrust vector in Cartesian coordinates. The thrust vector can be represented equivalently, using the thrust direction and the thrust modulation parameter σ [Eq. (3)] as control variables. With this new representation of the control, the thrust constraint is transformed into an inequality constraint on the thrust modulation parameter, given in the rightmost inequality of Eq. (3):

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$$0 \leq \|\mathbf{T}\| \leq T_{\text{MAX}} \Leftrightarrow \mathbf{T} = \sigma T_{\text{MAX}}\mathbf{e}, \begin{cases} 0 \leq \sigma \leq 1 \\ \|\mathbf{e}\| = 1 \end{cases} \tag{3}$$

From the Hamiltonian, and using the Pontryagin minimum principle [1], the first-order optimality conditions on the control are derived [Eq. (4)]. The inequality constraint on the thrust modulation parameter results in the bang–bang structure of the control defined by the switching function H_σ. The assumption is that there are no singular arcs; that is, the switching function is zero at isolated points, which is usually the case for interplanetary trajectories:

$$\mathbf{e} = -\boldsymbol{\lambda}_\mathbf{v}/\|\boldsymbol{\lambda}_\mathbf{v}\| \qquad \begin{cases} \sigma = 0 & \text{if } H_\sigma \geq 0 \\ \sigma = 1 & \text{if } H_\sigma < 0 \end{cases}$$
$$H_\sigma = -T_{\text{MAX}}(\|\boldsymbol{\lambda}_\mathbf{v}\|/m + \lambda_m/I_{\text{SP}}g_0) \tag{4}$$

The adjoint dynamic equation (5) is derived from the derivative of the Hamiltonian with respect to the state, using the optimal control of Eq. (4):

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{Bmatrix} \boldsymbol{\lambda}_\mathbf{r} \\ \boldsymbol{\lambda}_\mathbf{v} \\ \lambda_m \end{Bmatrix} = -\frac{\partial H}{\partial \mathbf{x}} = \begin{Bmatrix} (\mu/r^3)[\boldsymbol{\lambda}_\mathbf{v} - (3/r^2)(\boldsymbol{\lambda}_\mathbf{v}\cdot\mathbf{r})\mathbf{r}] \\ \boldsymbol{\lambda}_\mathbf{r} \\ -\sigma T_{\text{MAX}}\|\boldsymbol{\lambda}_\mathbf{v}\|/m^2 \end{Bmatrix} \tag{5}$$

The state vector at the initial time is completely specified [Eq. (6)], as is the position and velocity at the final time [Eq. (7)]. The final mass is free, since the cost function depends on it [Eq. (1)]; therefore, the final mass adjoint is specified [Eq. (8)]:

$$\mathbf{x}(t_0) = \begin{Bmatrix} \mathbf{r_0} \\ \mathbf{v_0} \\ m_0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \boldsymbol{\lambda}_\mathbf{r}(t_0) \\ \boldsymbol{\lambda}_\mathbf{v}(t_0) \\ \lambda_m(t_0) \end{Bmatrix}\text{free} \tag{6}$$

$$\begin{Bmatrix} \mathbf{r}(t_f) = \mathbf{r_f} \\ \mathbf{v}(t_f) = \mathbf{v_f} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \boldsymbol{\lambda}_\mathbf{r}(t_f) \\ \boldsymbol{\lambda}_\mathbf{v}(t_f) \end{Bmatrix}\text{free} \tag{7}$$

$$m(t_f)\text{free} \Rightarrow \lambda_m(t_f) = \frac{\partial \phi}{\partial m(t_f)} = -1 \tag{8}$$

The trajectory optimization problem has been reduced to a TPBVP, which is a system of nonlinear equations. The TPBVP consists of finding the initial adjoint values [Eq. (6)] that fulfill the final boundary constraints [Eqs. (7) and (8)], the dynamics defined in Eq. (2) and (7), and the optimal control given in Eq. (4). This TPBVP is invariant under the change of the adjoint scale. This property is used to reduce the dimension of the search space. A grid of a unit six sphere [Eq. (9)] is used to perform a systematic search of the parameter space. The most promising initial costate values obtained in this global exploration will then be used as an initial guess of a local search algorithm.

Because of the adjoint scale invariance of the problem, the final condition on the mass adjoint can be substituted with an inequality constraint [$\lambda_m(t_f) < 0$]. The mass adjoint dynamics given in Eq. (5) make the mass costate a monotonically decreasing function; hence, this condition is not very constraining.

An adjoint-control transformation that maps the initial adjoint values with a set of control-related variables was introduced [12] and later extended [9]. This adjoint-control transformation provides physical meaning and bounds to some (but not all) optimization parameters. The main advantages of the new transformation are the entirely bounded search space and its reduction in one dimension. These characteristics permit an efficient global exploration of the optimization space:

$$\begin{cases} \lambda_1 = \cos\varphi_1 \\ \lambda_2 = \sin\varphi_1\cos\varphi_2 \\ \lambda_3 = \sin\varphi_1\sin\varphi_2\cos\varphi_3 \\ \lambda_4 = \sin\varphi_1\sin\varphi_2\sin\varphi_3\cos\varphi_4 & ; \quad \varphi_k \in [0,\pi] \\ \lambda_5 = \sin\varphi_1\sin\varphi_2\sin\varphi_3\sin\varphi_4\cos\varphi_5 \\ \lambda_6 = \sin\varphi_1\sin\varphi_2\sin\varphi_3\sin\varphi_4\sin\varphi_5\cos\varphi_6 \\ \lambda_7 = \sin\varphi_1\sin\varphi_2\sin\varphi_3\sin\varphi_4\sin\varphi_5\sin\varphi_6 \end{cases}$$
$$k = 1,5; \qquad \varphi_6 \in [0, 2\pi] \tag{9}$$

The initial costate values selected during the global search are used to start a local search to solve the TPBVP (multistart technique). The algorithm selected to solve the TPBVP system of nonlinear equations is the trust-region dogleg algorithm available in [13] (fsolve). The discontinuities of the bang–bang optimal control prevent the convergence of this gradient-based solver unless the initial guess is very close to the final solution. A smoothing of the thrust modulation factor is implemented to enlarge the radius of convergence of the solver. Different smoothing techniques have been analyzed to solve the TPBVP [10]. The smoothed thrust modulation parameter is given in Eq. (10):

$$\sigma = 1/[1 + \exp(\varepsilon H_\sigma)], \qquad \varepsilon > 0 \tag{10}$$

In [10], a continuation method in the smoothing parameter was implemented in combination with the shooting method to solve the TPBVP. In the present method, the continuation on the smoothing parameter is not required. The smoothing parameter is selected to produce a quasi-bang–bang profile, but it is smooth enough to permit the gradient-based local optimizer to converge. For use in guidance schemes, a final refinement of the optimal solutions is sometimes performed, increasing the smoothing parameter by one order of magnitude to achieve a sharper bang–bang thrusting structure. The final mass obtained with the highest smoothing parameter ε can vary slightly due to the sharper thrust profile.

III. Validation

The first test scenario is an Earth–Venus transfer from [10]. The launch date was 7 October 2005, and the flight time was 2.73 years. The initial mass was 1500 kg, the maximum thrust was 0.33 N, and the specific impulse was 3800 s. The systematic search took 30 min. in a dual-core laptop at 2.4 GHz and evaluated almost 43,000 trajectories. The 20 best trajectories were selected as initial guesses for the local optimization. In the local optimization, each feasible trajectory took an average of 1 min. In [10], one global optimal s and two local optimal solutions, s_1 and s_2, were reported. The presented

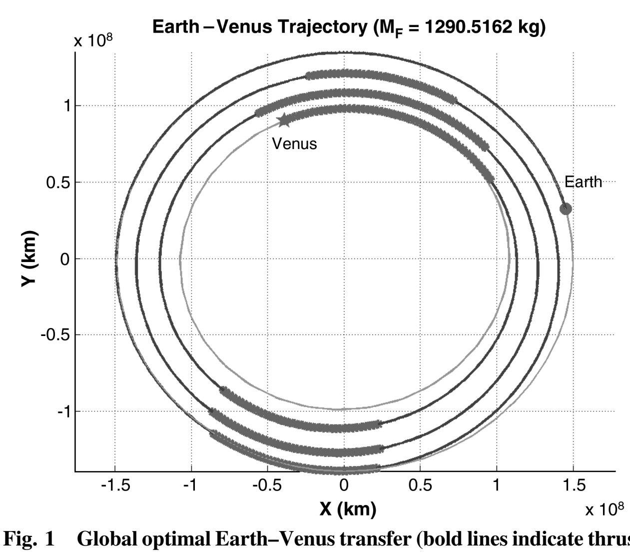

Fig. 1 Global optimal Earth–Venus transfer (bold lines indicate thrust arcs).

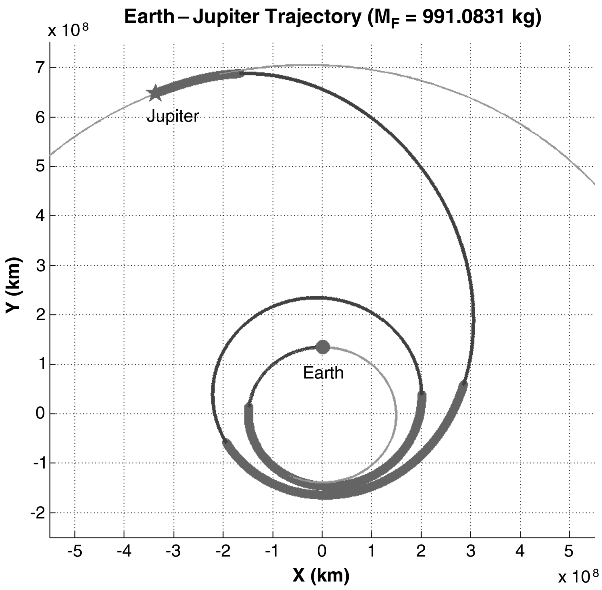

Fig. 2 Global optimal Earth–Jupiter transfer.

method found s seven times (Fig. 1), s_1 six times, s_2 twice, and an additional solution, s_3, once (there are four nonconvergent cases).

A sparse grid was configured that evaluated 1242 trajectories and retained 15 initial guessed for local optimization. The numerical propagation step size was increased to 7 days. The systematic search took 49 s. The solution s was obtained three times: s_1 twice and 10 nonconvergent cases. To obtain the best solution in a short time, the size of the grid and the number of initial guesses to be locally optimized should be selected properly. These parameters should be analyzed in a case-by-case basis to gain confidence about the global optimality of the solutions.

The next validation scenario is a direct Earth–Jupiter transfer from [11]. The initial mass was 1500 kg, the maximum thrust was 0.33 N, and the specific impulse was 3790 s. The departure date was December 21. The flight time was evaluated in a grid between 1965 and 2820 days. Only three nodes in the flight time grid were considered, because the purpose was to validate the method and its implementation. Different arrival dates had no impact on the computational time of the systematic search.

Two optimal trajectories are reported in [11]: s_1 with a flight time 1965 days and s_2 with a flight time 2820 days. These optimal trajectories and additional local optima are found (s_1 is depicted in Fig. 2). The final mass of each trajectory as a function of the flight time is depicted in Fig. 3. This figure shows the expected trend of the final mass variation with the flight time; optimal solutions with a

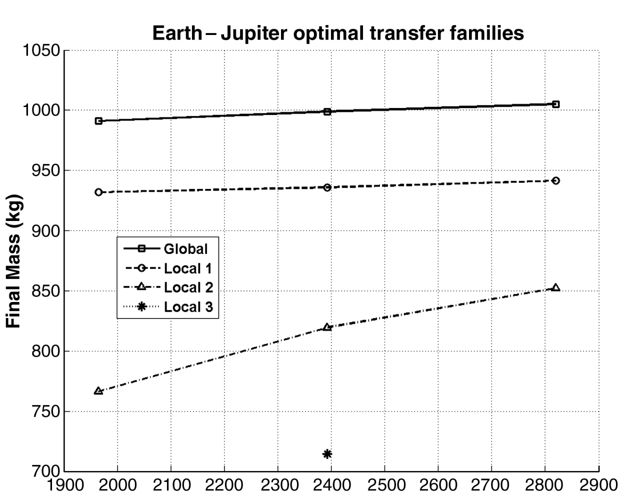

Fig. 3 Evolution of the final mass with the flight time.

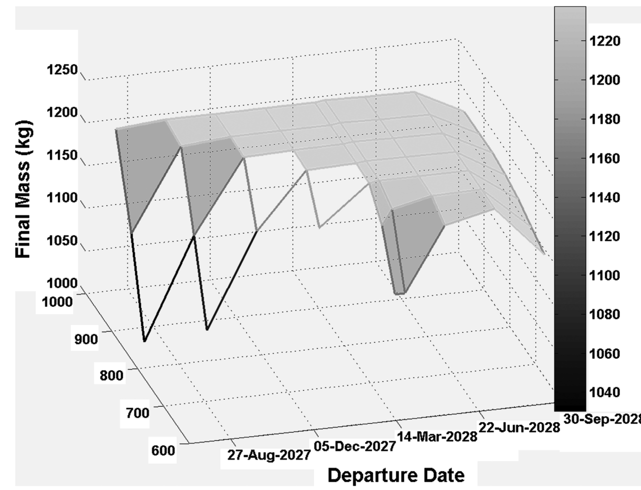

Fig. 4 Optimal Earth–Mars transfers.

longer flight time have a higher final mass for the same departure date.

The last benchmark scenario is an Earth–Mars direct transfer, described in [4]. The departure date was free between 16 July 2027 and 8 January 2029, and the flight time was free within 500 and 910 days. The initial mass was 1500 kg, the maximum thrust was 0.2 N, and the specific impulse was 3000 s. Two optimal trajectories are presented in [4], corresponding to different initial guesses. The reported final masses are 1236.8 and 1237.5 kg. It is worth mentioning that the latest had 1066 days of flight time, which is out of the 910-day limit proposed in [4].

With the same configuration parameters as previous scenarios, the optimization method found several solutions for each combination of launch date and flight time. The systematic search took about 10 min. per departure date. The nonlinear equation solver took about 1 min. per trajectory. The final mass of the optimal trajectories is presented in Fig. 4 as a function of the departure date and the flight time. The best optimal trajectory found is presented in Fig. 5. This trajectory has the same final mass (1237.5 kg) as the best solution reported in [4], but the flight time is 216 days shorter. Figure 4 shows that sharp variations in the final mass occur at certain departure dates and flight times (similar to the pork-chop plots of the impulsive case) when the global optimal solution is no longer feasible.

IV. Application to Guidance of BepiColombo

The presented method was applied as an academic exercise to the guidance problem of BepiColombo [14]. The BepiColombo mission

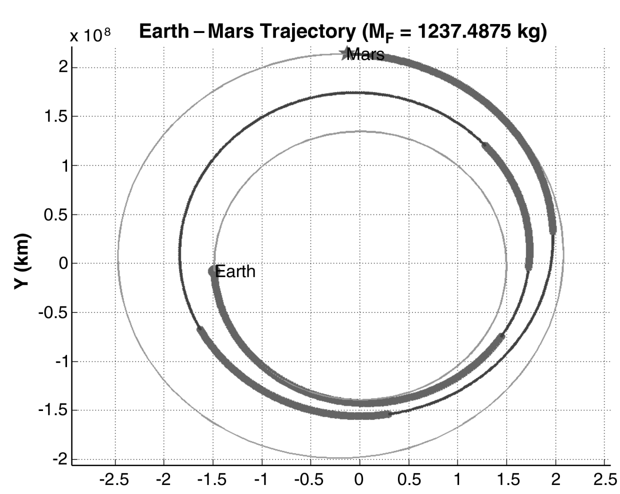

Fig. 5 Global optimal Earth–Mars transfer (departure on 24 March 2008, flight time 850 days).

aims at Mercury using solar electric propulsion and numerous gravity assists with Earth, Venus, and Mercury. The maximum available thrust depends on the heliocentric distance. The thrust solar aspect angle (SAA) is constrained within some limits ($\cos \mathrm{SAA}_{\mathrm{MAX}} \leq \mathbf{e} \cdot \mathbf{r}/r \leq \cos \mathrm{SAA}_{\mathrm{MIN}}$) that also vary with the heliocentric distance. The control direction constraint is included in the formulation of the optimal control using the Pontryagin minimum principle, as in Eq. (4). After departure from, and before arrival to, a planet there must be coast arcs (no thrust) of a few weeks duration.

The BepiColombo nominal trajectory is very complex and difficult to optimize [15]. The reoptimized arcs will provide the reference trajectory and thrust profile for the guidance and control functions, fulfilling all the above-mentioned constraints. These functions compensate the deviations introduced by navigation uncertainties, maneuver execution errors, and dynamics perturbations [16]. To provide some margin to the guidance and control, the maximum thrust level is reduced by 5%, and the thrust SAA limits are reduced by 3°. Thus, the reoptimized arcs do not have active constraints and provide a good starting point for the guidance and control.

One of the most interesting arcs connects the second Mercury gravity assist with the third one. The reoptimized trajectory is depicted in Fig. 6; the initial and final mandatory coast arcs are not depicted. The optimal thrust profile is depicted in Fig. 7, showing the 5% margin in the maximum thrust. The thrust SAA is depicted in Fig. 8, where the upper and lower constraints include a margin of 3°. This reoptimization can be used at any point in the trajectory if the deviation from the nominal trajectory is so large that it prevents the guidance and control to achieve the next encounter conditions.

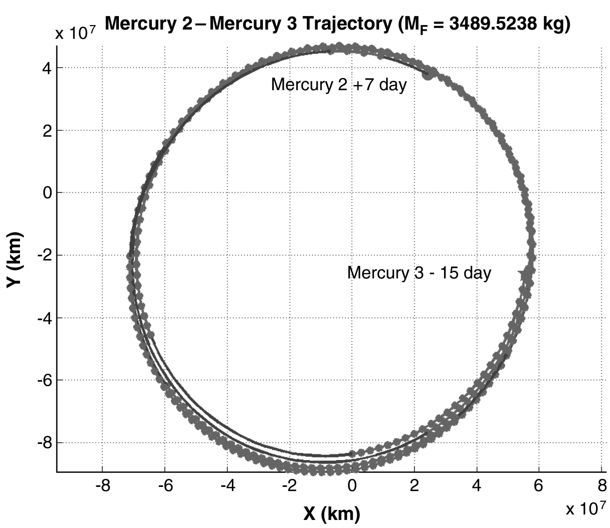

Fig. 6 Mercury 2–Mercury 3 reoptimized trajectory.

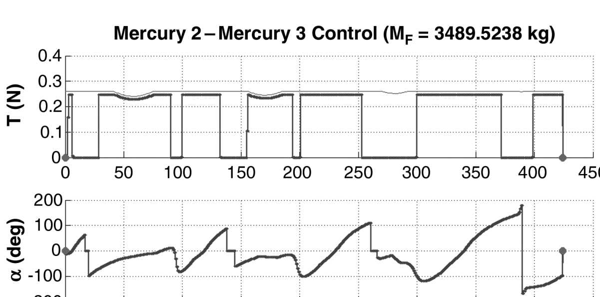

Fig. 7 Mercury 2–Mercury 3 optimal thrust profile.

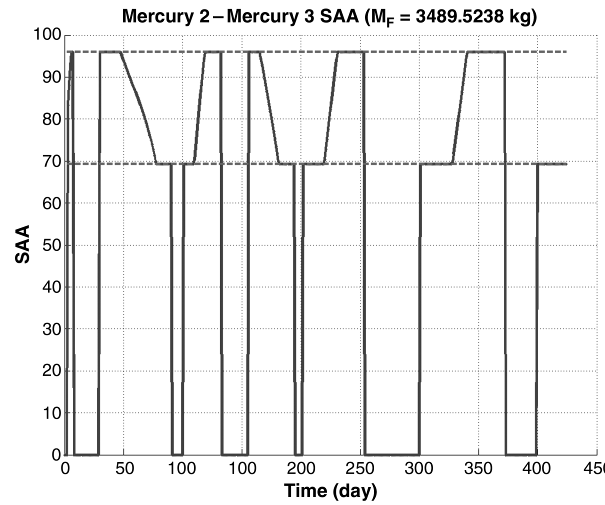

Fig. 8 Mercury 2–Mercury 3 thrust SAA.

V. Conclusions

An initial guess-free method for optimization of low-thrust transfers has been presented. The TPBVP is solved with an indirect shooting method in two steps. First, a mapping of the initial costates to a unit six sphere is done to conduct a systematic search on the optimization space. The new optimization space is bounded and has one dimension less than the seven-dimensional adjoint space. Then, a local search is performed on some selected candidates using a smoothing technique to increase the radius of convergence. The main benefit of the presented method is the fast optimization time and the few configuration parameters. The method is intended for aiding the mission analyst in early mission design, similar to the Lambert's problem solver in the impulsive case. The optimization method has been validated against several examples from the literature. The presented algorithm is part of a guidance scheme to reoptimize transfers fulfilling the operational constraints. It is being embedded in a hybrid method that optimizes multipoint boundary value problems to provide initial guesses.

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